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# Remarks on spectral gaps on the Riemannian path space

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## Abstract

In this paper, we will give some remarks on links between the spectral gap of the Ornstein-Uhlenbeck operator on the Riemannian path space with lower and upper bounds of the Ricci curvature on the base manifold; this work was motivated by a recent work of A. Naber on the characterization of the bound of the Ricci curvature by analysis of path spaces.

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## 1 Introduction

Let  $M$  be a complete smooth Riemannian manifold of dimension  $d$ , and  $Z$  a  $C^1$ -vector field on  $M$ . We will be concerned with the diffusion operator

$$L = \frac{1}{2}(\Delta_M - Z),$$

where  $\Delta_M$  is the Beltrami-Laplace operator on  $M$ . Let  $\nabla$  be the Levi-Civita connection and  $\text{Ric}$  the Ricci curvature tensor on  $M$ . We will denote

$$\text{Ric}_Z = \text{Ric} + \nabla Z.$$

It is well-known that the lower bound  $K_2$  of the symmetrized  $\text{Ric}_Z^s$ , that is,

$$\text{Ric}_Z^s(x) = \frac{1}{2}(\text{Ric}_Z(x) + \text{Ric}_Z^*(x)) \geq K_2 \text{Id}, \quad (1.1)$$

where  $\text{Ric}_Z^*$  denotes the transposed matrix of  $\text{Ric}_Z$ , gives the lower bound of constants in the logarithmic Sobolev inequality with respect to the heat measure  $\rho_t(x, dy)$ , associated to  $L$ ; more precisely,

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$$\int_M u^2(y) \log\left(\frac{u^2(y)}{\|u\|_{\rho_t}^2}\right) \rho_t(x, dy) \leq 2 \frac{1 - e^{-K_2 t}}{K_2} \int_M |\nabla u(y)|^2 \rho_t(x, dy), \quad t > 0, \quad (1.2)$$

where  $\|u\|_{\rho_t}^2 = \int_M u^2(y) \rho_t(x, dy)$ .

Given now a finite number of times  $0 < t_1 < \dots < t_N$ , consider the probability measure  $\nu_{t_1, \dots, t_N}$  on  $M^N$  defined by

$$\int_{M^N} f d\nu_{t_1, \dots, t_N} = \int_{M^N} f(y_1, \dots, y_N) p_{t_1}(x, dy_1) p_{t_2 - t_1}(y_1, dy_2) \cdots p_{t_N - t_{N-1}}(y_{N-1}, dy_N) \quad (1.3)$$

where  $f$  is a bounded measurable function on  $M^N$ . Then with respect to the correlated metric  $|\cdot|_C$  on  $TM^N$  (see definition (1.10) below), the logarithmic Sobolev inequality still holds for  $\nu_{t_1, \dots, t_N}$ , that is, there is a constant  $C_N > 0$  such that

$$\int_{M^N} f^2 \log\left(\frac{f^2}{\|f\|_{\nu_{t_1, \dots, t_N}}^2}\right) d\nu_{t_1, \dots, t_N} \leq C_N \int_{M^N} |\nabla f|_C^2 d\nu_{t_1, \dots, t_N}, \quad f \in C^1(M^N). \quad (1.4)$$

It was proved in [20, 6] that under the hypothesis

$$\sup_{x \in M} |||\text{Ric}_Z(x)||| < +\infty, \quad (1.5)$$

where  $|||\cdot|||$  denotes the norm of matrices, the constant  $C_N$  in (1.4) can be bounded, that is

$$\sup_{N \geq 1} C_N < +\infty. \quad (1.6)$$

A natural question is whether (1.6) still holds only under Condition (1.1)? In a recent work [21], A. Naber proved that if the uniform bound (1.6) holds, then the Ricci curvature of the base manifold has an upper bound. It is well-known that Inequality (1.2) implies the lower bound (1.1), therefore Condition (1.6) implies (1.5). The main purpose in [21] is to get informations on  $\text{Ric}_Z$  from the analysis of the Riemannian path space. Let's explain briefly the context.

Let  $O(M)$  be the bundle of orthonormal frames and  $\pi : O(M) \rightarrow M$  the canonical projection. Let  $H_1, \dots, H_d$  be the canonical horizontal vector fields on  $O(M)$ , consider the Stratanovich stochastic differential equation (SDE) on  $O(M)$ :

$$du_t(w) = \sum_{i=1}^d H_i(u_t(w)) \circ dw_t^i - \frac{1}{2} H_Z(u_t(w)) dt, \quad u_0(w) = u_0 \in \pi^{-1}(x), \quad (1.7)$$

where  $H_Z$  denotes the horizontal lift of  $Z$  to  $O(M)$ , that is,  $\pi'(u) \cdot H_Z(u) = Z(\pi(u))$ . It is well-known that under Condition (1.1), the life-time  $\tau_x$  of the SDE (1.7) is infinite. Let

$$\gamma_t(w) = \pi(u_t(w)). \quad (1.8)$$

Then  $\{\gamma_t(w); t \geq 0\}$  is a diffusion process on  $M$ , having  $L$  as generator. The probability measure  $\nu_{t_1, \dots, t_N}$  considered in (1.3) is the law of  $w \rightarrow (\gamma_{t_1}(w), \dots, \gamma_{t_N}(w))$  on  $M^N$ . Now consider the following path space

$$W_x^T(M) = \{\gamma : [0, T] \rightarrow M \text{ continuous, } \gamma(0) = x\}.$$

The law  $\mu_{x,T}$  on  $W_x^T(M)$  of  $w \rightarrow \gamma(w)$  is called the Wiener measure on  $W_x^T(M)$ . The integration by parts formula for  $\mu_{x,T}$  was first established in the Seminal book [5], then developed in [16, 10]; the Cameron-Martin type quasi-invariance of  $\mu_{x,T}$  was first proved by B. Driver [9], completed and simplified in [18, 19, 13]. By means of Cameron-Martin, we consider the space

$$\mathbb{H} = \left\{ h : [0, T] \rightarrow \mathbb{R}^d \text{ absolutely continuous; } h(0) = 0, |h|_{\mathbb{H}}^2 = \int_0^T |\dot{h}(s)|_{\mathbb{R}^d}^2 ds < +\infty \right\}$$

where the dot denotes the derivative with respect to the time  $t$ . Let  $F : W_x^T(M) \rightarrow \mathbb{R}$  be a cylindrical function in the form:  $F(\gamma) = f(\gamma(t_1), \dots, \gamma(t_N))$  for some  $N \geq 1, 0 \leq t_1 < t_2 < \dots < t_N \leq 1$ , and  $f \in C_b^1(M^N)$ . The usual gradient of  $F$  in Malliavin calculus is defined by

$$D_\tau F(\gamma(w)) = \sum_{j=1}^N u_{t_j}(w)^{-1} (\partial_j f)(\gamma_{t_1}(w), \dots, \gamma_{t_N}(w)) \mathbf{1}_{(\tau \leq t_j)}, \quad (1.9)$$

where  $\partial_j$  is the gradient with respect to the  $j$ -th component. The correlated norm of  $\nabla f$  is

$$|\nabla f|_C^2 = \sum_{j,k=1}^N \langle u_{t_j}(w)^{-1} (\partial_j f), u_{t_k}(w)^{-1} (\partial_k f) \rangle t_j \wedge t_k, \quad (1.10)$$

where  $t_j \wedge t_k$  denotes the minimum between  $t_j$  and  $t_k$ . Notice that the norm  $|\nabla f|_C$  is random. The generator  $\mathcal{L}_T^x$  associated to the Dirichlet form

$$\mathcal{E}(F, F) = \int_{W_x^T(M)} \left( \int_0^T |D_\tau F|^2(\gamma) d\tau \right) d\mu_{x,T}(\gamma)$$

is called the Ornstein-Uhlenbeck operator. The powerful tool of  $\Gamma_2$  of Bakry and Emery [3] is not applicable to  $\mathcal{L}_T^x$ , the reason for this is the geometry of  $W_x^T(M)$  inherited from  $\mathbb{H}$  is quite complicated, the associated ‘‘Ricci tensor’’ being a divergent object (see [7, 8, 12]). When the base manifold  $M$  is compact, the existence of the spectral gap for  $\mathcal{L}_T^x$  has been proved in [14]. The logarithmic Sobolev inequality for  $D_\tau F$  defined in (1.9) has been established in [2], as well as in [20] or [6] where the constant was estimated using the bound of Ricci curvature tensor of the base manifold  $M$ . The method used in [14] is the martingale representation, which takes advantage the Itô filtration; this method has been developed in [12] to deal with the problem of vanishing of harmonic forms on  $W_x^T(M)$ . The purpose in [21] is to proceed in the opposite direction, to get the bound for Ricci curvature tensor of the base manifold  $M$  from the analysis of the path space  $W_x^T(M)$ .

The organization of the paper is as follows. In section 2, we will recall briefly basic objects in Analysis of  $W_x^T(M)$ . On the path space  $W_x^T(M)$ , there exist two type of gradients: the usual one is more related to the geometry of the base manifold, while the damped one is easy to be handled. In section 3, we will make estimation of the spectral gap of  $\mathcal{L}_T^x$  as explicitly as possible in function of lower bound  $K_2$  and upper bound  $K_1$  of Ric. In section 4, we will study the behaviour of the spectral gap  $Spect(\mathcal{L}_T^x)$  as  $T \rightarrow 0$ . Roughly speaking, we will get the following result:

$$1 - \frac{K_1 T}{2} + o(T) \leq Spect(\mathcal{L}_T^x) \leq 1 + \frac{K_2 T}{2} + o(T), \quad \text{as } T \rightarrow 0$$

under the following condition (3.1).

## 2 Framework of the Riemannian path space

We shall keep the notations of Section 1, and throughout this section,  $u_t(w)$  denotes always the solution of (1.7) and  $\gamma_t(w)$  the path defined in (1.8). For any  $h \in \mathbb{H}$ , we introduce first the usual gradient on the path space  $W_x^T(M)$ , which gives Formula (1.9) when the functional  $F$  is a cylindrical function. To this end, let

$$q(t, h) = \int_0^t \Omega_{u_s(w)} \left( h(s), \circ dw(s) - \frac{1}{2} u_s(w)^{-1} Z_{\gamma_s(w)} ds \right) \quad (2.1)$$

where  $\Omega_u$  is the equivariant representation of the curvature tensor on  $M$ . Let  $\text{ric}_Z$  be the equivariant representation of  $\text{Ric}_Z$ , that is,

$$\text{ric}_Z(u) = u^{-1} \circ \text{Ric}_Z(\pi(u)) \circ u, \quad u \in O(M).$$

Consider  $\hat{h}(w) \in \mathbb{H}$  defined by

$$\dot{\hat{h}}_t(w) = \dot{h}(t) + \frac{1}{2} \text{ric}_Z(u_t(w)) h(t). \quad (2.2)$$

Let  $F : W_x^T(M) \rightarrow \mathbb{R}$  be a functional, we denote  $\tilde{F}(w) = F(\gamma.(w))$ . Then according to [16], we define

$$(D_h F)(\gamma.(w)) = \left\{ \frac{d}{d\varepsilon} \tilde{F} \left( \int_0^\cdot e^{\varepsilon q(s, h)} dw(s) + \varepsilon \hat{h} \right) \right\}_{\varepsilon=0}. \quad (2.3)$$

By [5, 16], if  $F$  is a cylindrical function on  $W_x^T(M)$ , then

$$(D_h F)(\gamma.(w)) = \int_0^T \langle D_\tau F(\gamma.(w)), \dot{h}(\tau) \rangle d\tau$$

where  $D_\tau F$  was given in (1.9). Consider the following resolvent equation

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2} \text{ric}_Z(u_t(w)) Q_{t,s}, \quad t \geq s, \quad Q_{s,s} = \text{Id}. \quad (2.4)$$

For a cylindrical function  $F$  on  $W_x^T(M)$  given by  $F(\gamma) = f(\gamma(t_1), \dots, \gamma(t_N))$  with  $f \in C_b^1(M^N)$ , following [16], we define the damped gradient  $\tilde{D}_\tau F$  of  $F$  by

$$\tilde{D}_\tau F(\gamma.(w)) = \sum_{j=1}^N Q_{t_j, \tau}^* (u_{t_j}(w)^{-1} \partial_j f) \mathbf{1}_{(\tau \leq t_j)}, \quad (2.5)$$

where  $Q_{\tau,s}^*$  is the transpose matrix of  $Q_{\tau,s}$ . The damped gradient  $\tilde{D}_\tau F$  on the path space  $W_x^T(M)$  plays a basic role in Analysis of  $W_x^T(M)$ . Let  $(v_t)_{t \geq 0}$  be a  $\mathbb{R}^d$ -valued process, adapted to the Itô filtration  $\mathcal{F}_t$  generated by  $\{w(s); s \leq t\}$  such that  $\mathbb{E}(\int_0^T |v_t|^2 dt) < +\infty$ . Consider two maps  $v \rightarrow \tilde{v}$  and  $v \rightarrow \hat{v}$  defined respectively by

$$\tilde{v}_t = v_t - \frac{1}{2} \text{ric}_{u_t(w)} \int_0^t Q_{t,s} v_s ds, \quad (2.6)$$

and

$$\hat{v}_t = v_t + \frac{1}{2} \text{ric}_{u_t(w)} \int_0^t v_s ds. \quad (2.7)$$

Then  $\hat{v} = \tilde{v} = v$ . The two gradients  $D_t F$  and  $\tilde{D}_t F$  are linked by the following formula

$$\int_0^T \langle \tilde{D}_t F, v_t \rangle dt = \int_0^T \langle D_t F, \tilde{v}_t \rangle dt. \quad (2.8)$$

The good feather of the damped gradient is that it admits a nice martingale representation

$$F = \mathbb{E}(F) + \int_0^T \langle \mathbb{E}^{\mathcal{F}_t}(\tilde{D}_t F), dw_t \rangle$$

where  $\mathbb{E}^{\mathcal{F}_t}$  denotes the conditional expectation with respect to  $\mathcal{F}_t$ . The following logarithmic Sobolev inequality holds ([11, 17]):

$$\mathbb{E} \left( F^2 \log \frac{F^2}{\|F\|_{L^2}^2} \right) \leq 2 \mathbb{E} \left( \int_0^T |\tilde{D}_t F|^2 dt \right). \quad (2.9)$$

### 3 Precise lower bound on the spectral gap

The inconvenient of Inequality (2.9) is that the geometric information of the base manifold  $M$  is completely hidden. Now we use the usual gradient  $D_t F$  to make involving the geometry of  $M$ . By (2.9), the matter is now to estimate  $\int_0^T |\tilde{D}_t F|^2 dt$  by  $|D_t F|^2$ . We assume that

$$K_2 \text{Id} \leq \text{ric}_Z^s, \quad |||\text{ric}_Z||| \leq K_1 \quad (3.1)$$

for two constants  $K_1, K_2$  with  $K_1 \geq 0$  and  $K_1 + K_2 \geq 0$ .

**Theorem 3.1.** *Let  $0 < t \leq T$ . Set*

$$\begin{aligned} \Lambda(t, T) = & 1 + \frac{K_1}{K_2} \left( 1 - e^{-\frac{K_2(T-t)}{2}} \right) + \frac{K_1}{K_2} \left( 1 - e^{-\frac{K_2 t}{2}} \right) \\ & + \left( \frac{K_1}{K_2} \right)^2 \left[ \left( 1 - e^{-\frac{K_2 t}{2}} \right) + \frac{1}{2} \left( e^{-\frac{K_2(T+t)}{2}} - e^{-\frac{K_2(T-t)}{2}} \right) \right]. \end{aligned} \quad (3.2)$$

*Then we have the relation:*

$$\int_0^T |\tilde{D}_t F|^2 dt \leq \int_0^T \Lambda(t, T) |D_t F|^2 dt. \quad (3.3)$$

**Proof.** From (2.5) and (2.8), we have

$$\tilde{D}_t F = D_t F - \frac{1}{2} \int_t^T Q_{s,t}^* \text{ric}_{u_s}^* D_s F ds. \quad (3.4)$$

Thus,

$$\begin{aligned} |\tilde{D}_t F|^2 = & |D_t F|^2 - \left\langle D_t F, \int_t^T Q_{s,t}^* \text{ric}_{u_s}^* D_s F ds \right\rangle + \frac{1}{4} \left| \int_t^T Q_{s,t}^* \text{ric}_{u_s}^* D_s F ds \right|^2 \\ := & I_1 + I_2 + I_3 \text{ respectively.} \end{aligned}$$

In the following we will estimate the term of  $I_2$  and  $I_3$ . Under the lower bound in (3.1),

$$|||Q_{s,t}^*||| \leq e^{-\frac{K_2(s-t)}{2}}, \quad s \geq t.$$

Let

$$\Lambda_1(t, T) := \int_t^T \left( e^{-\frac{K_2(s-t)}{4}} \right)^2 ds.$$

Then

$$\begin{aligned} |I_2| &\leq |D_t F| \int_t^T e^{-\frac{K_2(s-t)}{2}} K_1 |D_s F| ds \\ &\leq |D_t F| \sqrt{K_1 \int_t^T \left( e^{-\frac{K_2(s-t)}{4}} \right)^2 ds} \sqrt{K_1 \int_t^T e^{-\frac{K_2(s-t)}{2}} |D_s F|^2 ds} \\ &= |D_t F| \sqrt{K_1 \Lambda_1(t, T)} \sqrt{K_1 \int_t^T e^{-\frac{K_2(s-t)}{2}} |D_s F|^2 ds} \\ &\leq \frac{1}{2} \left\{ |D_t F|^2 K_1 \Lambda_1(t, T) + K_1 \int_t^T e^{-\frac{K_2(s-t)}{2}} |D_s F|^2 ds \right\}. \end{aligned}$$

and

$$\begin{aligned} |I_3| &\leq \frac{1}{4} \left| \int_t^T e^{-\frac{K_2(s-t)}{2}} K_1 |D_s F| ds \right|^2 \\ &\leq \frac{1}{4} K_1^2 \Lambda_1(t, T) \int_t^T e^{-\frac{K_2(s-t)}{2}} |D_s F|^2 ds. \end{aligned}$$

Combining all the above inequalities, we get

$$\begin{aligned} |\tilde{D}_t F|^2 &\leq \left( 1 + \frac{K_1}{2} \Lambda_1(t, T) \right) |D_t F|^2 + \left( 1 + \frac{K_1}{2} \Lambda_1(t, T) \right) \frac{K_1}{2} \int_t^T e^{-\frac{K_2(s-t)}{2}} |D_s F|^2 ds \\ &= \left( 1 + \frac{K_1}{2} \Lambda_1(t, T) \right) \left( |D_t F|^2 + \frac{K_1}{2} \int_t^T e^{-\frac{K_2(s-t)}{2}} |D_s F|^2 ds \right). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \int_0^T |\tilde{D}_t F|^2 dt &\leq \int_0^T \left( 1 + \frac{K_1}{2} \Lambda_1(t, T) \right) |D_t F|^2 dt \\ &\quad + \int_0^T \left( 1 + \frac{K_1}{2} \Lambda_1(t, T) \right) \frac{K_1}{2} \int_t^T e^{-\frac{K_2(s-t)}{2}} |D_s F|^2 ds dt \\ &= \int_0^T \left( 1 + \frac{K_1}{2} \Lambda_1(s, T) \right) |D_s F|^2 ds \\ &\quad + \int_0^T |D_s F|^2 ds \int_0^s \frac{K_1}{2} \left( 1 + \frac{K_1}{2} \Lambda_1(t, T) \right) e^{-\frac{K_2(s-t)}{2}} dt \\ &:= \int_0^T \left( 1 + \frac{K_1}{2} \Lambda_1(s, T) \right) |D_s F|^2 ds + \int_0^T (J_1(s) + J_2(s)) |D_s F|^2 ds, \end{aligned}$$

where

$$J_1(s) := \int_0^s \frac{K_1}{2} e^{-\frac{K_2(s-t)}{2}} dt, \quad J_2(s) := \int_0^s \left( \frac{K_1}{2} \right)^2 \Lambda_1(t, T) e^{-\frac{K_2(s-t)}{2}} dt.$$

Next, then we compute the term  $J_1(s)$  and  $J_2(s)$ . By direct computation, we have

$$J_1(s) = \frac{K_1}{K_2} \left( 1 - e^{-\frac{K_2 s}{2}} \right)$$

and

$$\begin{aligned}
J_2(s) &= \left(\frac{K_1}{2}\right)^2 \int_0^s \frac{2}{K_2} \left(1 - e^{-\frac{K_2(T-t)}{2}}\right) e^{-\frac{K_2(s-t)}{2}} dt \\
&= \left(\frac{K_1}{2}\right)^2 \frac{2}{K_2} \left[ \int_0^s e^{-\frac{K_2(s-t)}{2}} dt - e^{-\frac{K_2(T+s)}{2}} \int_0^s e^{K_2 t} dt \right] \\
&= \left(\frac{K_1}{2}\right)^2 \frac{2}{K_2} \left[ \frac{2}{K_2} \left(1 - e^{-\frac{K_2 s}{2}}\right) - \frac{1}{K_2} e^{-\frac{K_2(T+s)}{2}} (e^{K_2 s} - 1) \right] \\
&= \left(\frac{K_1}{2}\right)^2 \frac{2}{K_2} \left[ \frac{2}{K_2} \left(1 - e^{-\frac{K_2 s}{2}}\right) + \frac{1}{K_2} e^{-\frac{K_2(T+s)}{2}} - \frac{1}{K_2} e^{-\frac{K_2(T-s)}{2}} \right].
\end{aligned}$$

Adding  $J_1(s)$  to  $J_1(s)$  implying that

$$\begin{aligned}
&J_1(s) + J_2(s) \\
&= \frac{K_1}{K_2} \left(1 - e^{-\frac{K_2 s}{2}}\right) + \left(\frac{K_1}{K_2}\right)^2 \left[ \left(1 - e^{-\frac{K_2 s}{2}}\right) + \frac{1}{2} \left(e^{-\frac{K_2(T+s)}{2}} - e^{-\frac{K_2(T-s)}{2}}\right) \right] := \Lambda_2(s, T)
\end{aligned}$$

Thus,

$$\int_0^T |\tilde{D}_t F|^2 dt \leq \int_0^T \Lambda(t, T) |D_t F|^2 dt,$$

with

$$\begin{aligned}
\Lambda(t, T) &= 1 + \frac{K_1}{2} \Lambda_1(t, T) + \Lambda_2(t, T) \\
&= 1 + \frac{K_1}{K_2} \left(1 - e^{-\frac{K_2(T-t)}{2}}\right) + \frac{K_1}{K_2} \left(1 - e^{-\frac{K_2 t}{2}}\right) \\
&\quad + \left(\frac{K_1}{K_2}\right)^2 \left[ \left(1 - e^{-\frac{K_2 t}{2}}\right) + \frac{1}{2} \left(e^{-\frac{K_2(T+t)}{2}} - e^{-\frac{K_2(T-t)}{2}}\right) \right].
\end{aligned}$$

The proof is completed.  $\square$

Notice that as  $K_2 \rightarrow 0$ , by expression (3.2),

$$\Lambda(t, T) \rightarrow 1 + \frac{K_1 T}{2} + K_1^2 \left(\frac{Tt}{4} - \frac{t^2}{8}\right).$$

Now we study the variation of the function  $t \rightarrow \Lambda(t, T)$ . It is quite interesting to remark that its monotonicity is dependent of the sign of  $K_2$ .

**Proposition 3.2.** (i) If  $K_2 < 0$ , then  $t \rightarrow \Lambda(t, T)$  is strictly increasing over  $[0, T]$ . (ii) If  $K_2 > 0$ , then the maximum is attained at a point  $t_0$  in  $(0, T)$ .

**Proof.** Taking the derivative of  $t \rightarrow \Lambda(t, T)$  gives

$$\begin{aligned}
\Lambda'(t, T) &= -\frac{K_1}{2} e^{-\frac{K_2(T-t)}{2}} + \frac{K_1}{2} e^{-\frac{K_2 t}{2}} \\
&\quad + \frac{K_1^2}{2K_2} e^{-\frac{K_2 t}{2}} - \frac{K_1^2}{4K_2} e^{-\frac{K_2(T+t)}{2}} - \frac{K_1^2}{4K_2} e^{-\frac{K_2(T-t)}{2}}.
\end{aligned}$$

In addition, we have

$$\Lambda(0, T) = 1 + \frac{K_1}{K_2} \left(1 - e^{-\frac{K_2 T}{2}}\right)$$



and

$$\begin{aligned}
\Lambda(T, T) &= 1 + \frac{K_1}{K_2} \left(1 - e^{-\frac{K_2 T}{2}}\right) + \left(\frac{K_1}{K_2}\right)^2 \left[ \left(1 - e^{-\frac{K_2 T}{2}}\right) + \frac{1}{2} \left(e^{-K_2 T} - 1\right) \right] \\
&= 1 + \frac{K_1}{K_2} \left(1 - e^{-\frac{K_2 T}{2}}\right) + \frac{1}{2} \left(\frac{K_1}{K_2}\right)^2 \left(1 - e^{-\frac{K_2 T}{2}}\right)^2 \\
&= \frac{1}{2} + \frac{1}{2} \left[ 1 + \frac{K_1}{K_2} \left(1 - e^{-\frac{K_2 T}{2}}\right) \right]^2 = \frac{1}{2} + \frac{1}{2} \Lambda^2(0, T).
\end{aligned}$$

From the second equality in the above, we observe that  $\Lambda(T, T) \geq \Lambda(0, T)$ . Moreover,

$$\begin{aligned}
\Lambda'(0, T) &= -\frac{K_1}{2} e^{-\frac{K_2 T}{2}} + \frac{K_1}{2} + \frac{K_1^2}{2K_2} - \frac{K_1^2}{4K_2} e^{-\frac{K_2 T}{2}} - \frac{K_1^2}{4K_2} e^{-\frac{K_2 T}{2}} \\
&= \frac{K_1}{2} \left(1 - e^{-\frac{K_2 T}{2}}\right) + \frac{K_1^2}{2K_2} \left(1 - e^{-\frac{K_2 T}{2}}\right) \\
&= \frac{K_1}{2} (K_1 + K_2) \frac{1 - e^{-\frac{K_2 T}{2}}}{K_2} \geq 0;
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\Lambda'(T, T) &= -\frac{K_1}{2} + \frac{K_1}{2} e^{-\frac{K_2 T}{2}} + \frac{K_1^2}{2K_2} e^{-\frac{K_2 T}{2}} - \frac{K_1^2}{4K_2} e^{-K_2 T} - \frac{K_1^2}{4K_2} \\
&= -\frac{K_1}{2} + \frac{K_1}{2} e^{-\frac{K_2 T}{2}} - \frac{K_1^2}{4K_2} \left(1 - 2e^{-\frac{K_2 T}{2}} + e^{-K_2 T}\right) \\
&= -\frac{K_1}{2} \left(1 - e^{-\frac{K_2 T}{2}}\right) - \frac{K_1^2}{4K_2} \left(1 - e^{-\frac{K_2 T}{2}}\right)^2.
\end{aligned} \tag{3.6}$$

We see that

$$\begin{cases} \Lambda'(T, T) > 0 & \text{if } K_2 < 0, \\ \Lambda'(T, T) < 0 & \text{if } K_2 > 0. \end{cases} \tag{3.7}$$

Now we look for  $t \in [0, T]$  such that  $\Lambda'(t, T) = 0$ . We have

$$\begin{aligned}
\Lambda'(t, T) &= 0 \\
\Leftrightarrow \quad &\left(-\frac{K_1}{2} e^{-\frac{K_2 T}{2}} - \frac{K_1^2}{4K_2} e^{-\frac{K_2 T}{2}}\right) e^{K_2 t} + \left(\frac{K_1}{2} + \frac{K_1^2}{2K_2} - \frac{K_1^2}{4K_2} e^{-\frac{K_2 T}{2}}\right) = 0 \\
\Leftrightarrow \quad &-\frac{K_1}{4} e^{-\frac{K_2 T}{2}} \left(2 + \frac{K_1}{K_2}\right) e^{K_2 t} + \frac{K_1}{4} \left(2 + \frac{2K_1}{K_2} - \frac{K_1}{K_2} e^{-\frac{K_2 T}{2}}\right) = 0 \\
\Leftrightarrow \quad &e^{-\frac{K_2 T}{2}} \left(2 + \frac{K_1}{K_2}\right) e^{K_2 t} = \left(2 + \frac{2K_1}{K_2} - \frac{K_1}{K_2} e^{-\frac{K_2 T}{2}}\right).
\end{aligned} \tag{3.8}$$

Therefore there exists at most one  $t$  such that  $\Lambda'(t, T) = 0$ . For the case where  $K_2 < 0$ , if there exists  $t_0 \in (0, T)$  such that  $\Lambda(t_0, T) < 0$ . Then by (3.5) and (3.7), the equation  $\Lambda'(t, T) = 0$  has at least two solutions, it is impossible. Therefore for  $K_2 < 0$ ,  $\Lambda'(t, T) \geq 0$ . For  $K_2 > 0$ , we suppose  $t_0$  such that  $\Lambda'(t_0, T) = 0$ . Let  $\beta = \frac{K_1}{K_2}$ , then by (3.8)

$$e^{K_2 t_0} = \left(1 + \frac{\beta}{2 + \beta} \left(1 - e^{-\frac{K_2 T}{2}}\right)\right) e^{\frac{K_2 T}{2}},$$

or  $t_0 \in (0, T)$  is such that

$$e^{\frac{K_2 t_0}{2}} = \sqrt{1 + \frac{\beta}{2 + \beta} \left(1 - e^{-\frac{K_2 T}{2}}\right)} e^{\frac{K_2 T}{4}}. \tag{3.9}$$

The proof is completed.  $\square$

**Proposition 3.3.** Let  $\beta = \frac{K_1}{K_2}$ , then (i) if  $K_2 > 0$ ,

$$\begin{aligned} \sup_{t \in [0, T]} \Lambda(t, T) &= (1 + \beta)^2 - \left( \beta + \frac{\beta^2}{2} \right) \sqrt{1 + \frac{\beta}{2 + \beta} \left( 1 - e^{-\frac{K_2 T}{2}} \right)} e^{-\frac{K_2 T}{4}} \\ &\quad - \frac{\left( \beta + \beta^2 - \frac{\beta^2}{2} e^{-\frac{K_2 T}{2}} \right)}{\sqrt{1 + \frac{\beta}{2 + \beta} \left( 1 - e^{-\frac{K_2 T}{2}} \right)}} e^{-\frac{K_2 T}{4}}. \end{aligned} \quad (3.10)$$

(ii) if  $K_2 < 0$ ,

$$\sup_{t \in [0, T]} \Lambda(t, T) = \frac{1}{2} + \frac{1}{2} \left( 1 + \frac{K_1}{K_2} \left[ 1 - e^{-\frac{K_2 T}{2}} \right] \right)^2. \quad (3.11)$$

**Proof.** For  $K_2 > 0$ , we have

$$\begin{aligned} \Lambda(t_0, T) &= 1 + \beta \left( 1 - e^{-\frac{K_2 T}{2}} \cdot e^{\frac{K_2 t_0}{2}} \right) + \beta \left( 1 - e^{\frac{K_2 t_0}{2}} \right) \\ &\quad + \beta^2 \left[ \left( 1 - e^{-\frac{K_2 t_0}{2}} \right) + \frac{1}{2} \left( e^{-\frac{K_2 T}{2}} \cdot e^{\frac{K_2 t_0}{2}} - e^{-\frac{K_2 T}{2}} \cdot e^{\frac{K_2 t_0}{2}} \right) \right] \\ &= 1 + 2\beta + \beta^2 - \left( \beta + \frac{\beta^2}{2} \right) e^{-\frac{K_2 T}{2}} \cdot e^{\frac{K_2 t_0}{2}} - \left( \beta + \beta^2 - \frac{\beta^2}{2} e^{-\frac{K_2 T}{2}} \right) e^{-\frac{K_2 t_0}{2}}. \end{aligned}$$

Using (3.9) yields (3.10). For  $K_2 < 0$ ,  $\sup_{t \in [0, T]} \Lambda(t, T) = \Lambda(T, T)$ , which gives (3.11).  $\square$

Combining (2.9) and (3.3), we get

**Theorem 3.4.** Let  $C(T, K_1, K_2) = \sup_{t \in [0, T]} \Lambda(t, T)$ ; then it holds

$$\mathbb{E} \left( F^2 \log \frac{F^2}{\|F\|_{L^2}^2} \right) \leq 2C(T, K_1, K_2) \mathbb{E} \left( \int_0^T |D_t F|^2 dt \right) \quad (3.12)$$

for any cylindrical function  $F$  on  $W_x^T(M)$ .

It is well-known that the above logarithmic Sobolev inequality implies that the spectral gap of  $\mathcal{L}_T^x$ , denoted by  $Spect(\mathcal{L}_T^x)$ , has the following lower bound

$$Spect(\mathcal{L}_T^x) \geq \frac{1}{C(T, K_1, K_2)}.$$

**Theorem 3.5.** Assume (3.1) holds, then (i) if  $K_2 > 0$ , we have

$$Spect(\mathcal{L}_T^x)^{-1} \leq \left( 1 + \frac{K_1}{K_2} \right)^2 - \frac{K_1}{K_2} \sqrt{\left( 2 + \frac{K_1}{K_2} \right) \left( 2 + 2\frac{K_1}{K_2} - \frac{K_1}{K_2} e^{-\frac{K_2 T}{2}} \right)} e^{-\frac{K_2 T}{4}}; \quad (3.13)$$

(ii) if  $K_2 < 0$ , we have

$$Spect(\mathcal{L}_T^x)^{-1} \leq \frac{1}{2} + \frac{1}{2} \left( 1 + \frac{K_1}{K_2} \left[ 1 - e^{-\frac{K_2 T}{2}} \right] \right)^2. \quad (3.14)$$

**Proof.** Using the elementary inequality:  $A + B \geq 2\sqrt{AB}$  to the last two terms in (3.10) yields (3.13). Inequality (3.14) is obvious.  $\square$

It is quite interesting to remark that

**Proposition 3.6.** Let  $\psi(T, K_1, K_2)$  be the right hand side of (3.13) when  $K_2 > 0$  and the right hand side of (3.14) for  $K_2 < 0$ , then

$$\psi(T, K_1, K_2) \rightarrow 1 + \frac{K_1 T}{2} + \frac{K_1^2 T^2}{8} \quad \text{as } K_2 \rightarrow 0. \quad (3.15)$$

**Proof.** It is easy to see that the right hand side of (3.14) tends to  $1 + \frac{K_1 T}{2} + \frac{K_1^2 T^2}{8}$  as  $K_2 \rightarrow 0$ . For the right hand side of (3.13), we first remark that

$$(a) \quad \frac{K_1}{K_2} e^{-\frac{K_2 T}{4}} = \frac{K_1}{K_2} - \frac{K_1 T}{4} + \frac{K_1 K_2 T^2}{32} + o(K_2).$$

Secondly

$$\begin{aligned} & \left(2 + \frac{K_1}{K_2}\right) \left(2 + 2\frac{K_1}{K_2} - \frac{K_1}{K_2} e^{-\frac{K_2 T}{2}}\right) \\ &= \left(2 + \frac{K_1}{K_2}\right) \left(2 + \frac{K_1}{K_2} + \frac{K_1 T}{2} - \frac{K_1 K_2 T^2}{8} + o(K_2)\right) \\ &= \left(2 + \frac{K_1}{K_2}\right)^2 \left(1 + \frac{\frac{K_1 T}{2} - \frac{K_1 K_2 T^2}{8} + o(K_2)}{2 + \frac{K_1}{K_2}}\right) \end{aligned}$$

Therefore

$$\begin{aligned} & \sqrt{\left(2 + \frac{K_1}{K_2}\right) \left(2 + 2\frac{K_1}{K_2} - \frac{K_1}{K_2} e^{-\frac{K_2 T}{2}}\right)} \\ &= \left(2 + \frac{K_1}{K_2}\right) \left(1 + \frac{\frac{K_1 T}{2} - \frac{K_1 K_2 T^2}{8} + o(K_2)}{2 + \frac{K_1}{K_2}} - \frac{K_1 K_2 T^2}{32} + o(K_2^2)\right) \\ &= \left(2 + \frac{K_1}{K_2}\right) + \frac{K_1 T}{4} - \frac{3K_1 K_2 T^2}{32} + o(K_2). \end{aligned}$$

Combining this with (a), we get

$$\begin{aligned} & \frac{K_1}{K_2} e^{-\frac{K_2 T}{4}} \sqrt{\left(2 + \frac{K_1}{K_2}\right) \left(2 + 2\frac{K_1}{K_2} - \frac{K_1}{K_2} e^{-\frac{K_2 T}{2}}\right)} \\ &= \left(2 + \frac{K_1}{K_2}\right) \frac{K_1}{K_2} - \frac{K_1 T}{2} - \frac{K_1^2 T^2}{8} + o(K_2). \end{aligned}$$

Then (3.15) follows from the right hand side of (3.13).  $\square$

**Corollary 3.7.** Assume (3.1) holds.

(1) If  $K_1 = K_2 = K > 0$ , then

$$\psi(T, K, K) = 4 - \sqrt{3\left(4 - e^{-\frac{KT}{2}}\right)} e^{-\frac{KT}{4}} \rightarrow 1 \quad \text{as } K \rightarrow 0.$$

(2) If  $K_2 = -K_1 = -K$ , then

$$\psi(T, K, -K) = \frac{1}{2}(1 + e^{KT}).$$

**Remark.** Our results improve estimates obtained in [1].

## 4 Behaviour of $\text{Spect}(\mathcal{L}_T^x)$ as $T \rightarrow 0$

In this section, we consider the case where  $Z = 0$ . Then Condition (3.1) can be readed as

$$K_2 \text{Id} \leq \text{ric} \leq K_1 \text{Id}, \quad \text{with } K_1 + K_2 \geq 0 \quad (4.1)$$

and SDE (1.7) is reduced to

$$du_t(w) = \sum_{i=1}^d H_i(u_u(w)) \circ dw_t^i, \quad u_0(w) = u_0 \in \pi^{-1}(x). \quad (4.2)$$

The path  $\gamma_t(w) = \pi(u_t(w))$  is called Brownian motion path on  $M$ . Let  $\rho(x, y)$  be the Riemannian distance. By [22, p. 199], there is  $\varepsilon > 0$  such that

$$\sup_{t \in [0, T]} \mathbb{E} \left( \exp \left( \varepsilon \frac{\rho(x, \gamma_t)^2}{2t} \right) \right) < +\infty. \quad (4.3)$$

Assume that the curvature tensor satisfies the following growth condition

$$|||\Omega_u||| + \sum_{i=1}^d |||(L_{H_i}\Omega)_u||| \leq C (1 + \rho(x, \pi(u))^2) \quad (4.4)$$

where  $L_{H_i}$  denotes the Lie derivative with respect to  $H_i$ .

Let  $v \in \mathbb{H}$ , consider the functional  $F_T : W_x^T(M) \rightarrow \mathbb{R}$  defined by

$$F_T(\gamma(w)) = \int_0^T \langle \dot{v}(t), dw_t \rangle.$$

Let  $h \in \mathbb{H}$ ; then by (2.3), we have (see also [15])

$$(D_h F_T)(\gamma(w)) = \int_0^T \langle \dot{v}(t), q(t, h) dw_t \rangle + \int_0^T \langle \dot{v}(t), \dot{h}_t(w) \rangle dt. \quad (4.5)$$

Let  $a \in \mathbb{R}^d$  and consider  $v(t) = ta$  with  $|a| = 1$  in (4.5), we have

$$(D_h F_T)(\gamma(w)) = - \int_0^T \langle q(t, h)a, dw_t \rangle + \int_0^T \langle a, \dot{h}_t(w) \rangle dt. \quad (4.6)$$

Let  $\{e_1, \dots, e_d\}$  be an orthonormal basis of  $\mathbb{R}^d$ ; define

$$C_i(w, t, \tau) = - \int_\tau^t \Omega_{u_s(w)}(e_i, \circ dw(s)) \mathbf{1}_{(\tau < t)}.$$

Then by Fubini theorem, the term  $q(t, h)$  has the expression

$$q(t, h) = - \sum_{i=1}^d \int_0^T \dot{h}^i(\tau) C_i(w, t, \tau) d\tau.$$

According to (4.6), the gradient  $D_\tau F_T$  has the following expression:

$$(D_\tau F_T)(\gamma(w)) = \sum_{i=1}^d \left( \int_\tau^T \langle C_i(w, s, \tau)a, dw_s \rangle \right) e_i + a + \frac{1}{2} \int_\tau^T \text{ric}_Z(u_s) a ds. \quad (4.7)$$

We have

$$\text{Var}(F_T) = \mathbb{E}(F_T^2) - \mathbb{E}(F_T)^2 = |a|^2 T = T. \quad (4.8)$$

**Proposition 4.1.** Assume (4.4). Let

$$\chi_T = \frac{\mathbb{E}\left(\int_0^T |D_\tau F|^2 d\tau\right)}{\text{Var}(F_T)}.$$

Then

$$\chi_T = 1 + \frac{T}{2} \langle \text{ric}_Z(u_0)a, a \rangle + o(T) \quad \text{as } T \rightarrow 0 \quad (4.9)$$

where  $u_0$  is the initial point of (4.2).

**Proof.** We have, using (4.7),

$$\begin{aligned} |D_\tau F_T|^2 &= \sum_{i=1}^d \left( \int_\tau^T \langle C_i(w, s, \tau)a, dw_s \rangle \right)^2 + |a|^2 + \frac{1}{4} \left| \int_\tau^T \text{ric}(u_s)a ds \right|^2 \\ &\quad + \left\langle a, \int_\tau^T \text{ric}(u_s)a ds \right\rangle + 2 \sum_{i=1}^d \int_\tau^T \langle C_i(w, s, \tau)a, dw_s \rangle a^i \\ &\quad + 2 \int_0^d \int_\tau^T \langle C_i(w, s, \tau)a, dw_s \rangle \cdot \int_\tau^T \langle \text{ric}(u_s)a, e_i \rangle ds. \end{aligned}$$

Put respectively

$$\mathbb{E}\left(\int_0^T |D_\tau F_T|^2 d\tau\right) = I_1(T) + I_2(T) + I_3(T) + I_4(T) + I_5(T) + I_6(T).$$

It is obvious that  $I_2(T) = |a|^2 T = T$  and  $I_5(T) = 0$ . We have

$$I_1(T) = \sum_{i=1}^d \int_0^T \left( \int_\tau^T \mathbb{E}(|C_i(w, s, \tau)a|^2) ds \right) d\tau.$$

Now by growth condition (4.4) and (4.3), there is a constant  $\delta > 0$  such that

$$\mathbb{E}(|C_i(w, s, \tau)a|^2) \leq \delta(s - \tau). \quad (4.10)$$

So that  $I_1(T) \leq \delta T^3/6$ . By condition (4.1), it is easy to see that  $I_3(T) \leq \frac{K_1^2 T^3}{12}$ . It follows that  $I_6(T) \leq \frac{\sqrt{\delta} K_1}{6} T^3$ . Now for  $I_4(T)$ , we have

$$\lim_{T \rightarrow 0} \frac{I_4(T)}{T^2} = \frac{1}{2} \langle \text{ric}(u_0)a, a \rangle.$$

Combining these estimates together with (4.8), we get (4.9).  $\square$

**Theorem 4.2.** Assume (4.1) and (4.4). Let  $K_2(x)$  be the lower bound of  $\text{Ric}_x$ . Then as  $T \rightarrow 0$ ,

$$1 - \frac{K_1 T}{2} + o(T) \leq \text{Spect}(\mathcal{L}_T^x) \leq 1 + \frac{K_2(x)T}{2} + o(T). \quad (4.11)$$

**Proof.** For  $K_2 > 0$ , set  $\beta = \frac{K_1}{K_2}$ . As  $T \rightarrow 0$ , we have

$$\begin{aligned}
\sqrt{(2+\beta)\left(2+2\beta-\beta e^{-\frac{K_2 T}{2}}\right)} &= \sqrt{(2+\beta)^2\left(1+\frac{\beta}{2+\beta}\left(1-e^{-\frac{K_2 T}{2}}\right)\right)} \\
&= (2+\beta)\sqrt{1+\frac{\beta}{2+\beta}\frac{K_2 T}{2}+o(T)} \\
&= (2+\beta)\left(1+\frac{\beta}{2+\beta}\frac{K_2 T}{4}+o(T)\right).
\end{aligned}$$

So, for  $K_2 > 0$ , as  $T \rightarrow 0$ ,

$$\begin{aligned}
&\beta\sqrt{(2+\beta)\left(2+2\beta-\beta e^{-\frac{K_2 T}{2}}\right)} e^{-\frac{K_2 T}{4}} \\
&= \beta(2+\beta)\left(1+\frac{\beta}{2+\beta}\frac{K_2 T}{4}+o(T)\right)\left(1-\frac{K_2}{4}T+o(T)\right) \\
&= \beta(2+\beta)\left[1+\frac{T}{4}\left(\frac{K_1}{2+\beta}-K_2\right)+o(T)\right] \\
&= \beta(2+\beta)\left[1-\frac{K_2 T}{2(2+\beta)}+o(T)\right].
\end{aligned}$$

By (3.13), we get

$$Spect(\mathcal{L}_T^x)^{-1} \leq (1+\beta)^2 - \beta(2+\beta)\left[1-\frac{K_2 T}{2(2+\beta)}+o(T)\right] = 1 + \frac{K_1 T}{2} + o(T),$$

which implies that

$$Spect(\mathcal{L}_T^x) \geq 1 - \frac{K_1 T}{2} + o(T).$$

For  $K_2 < 0$ , by (3.14),

$$\begin{aligned}
Spect(\mathcal{L}_T^x)^{-1} &\leq \frac{1}{2} + \frac{1}{2}\left(1+K_1\frac{1-e^{-\frac{K_2 T}{2}}}{K_2}\right)^2 = \frac{1}{2} + \frac{1}{2}\left(1+\frac{K_1}{K_2}\left(\frac{K_2 T}{2}+o(T)\right)\right)^2 \\
&= 1 + \frac{K_1 T}{2} + o(T),
\end{aligned}$$

which implies again

$$Spect(\mathcal{L}_T^x) \geq 1 - \frac{K_1 T}{2} + o(T).$$

Now in (4.9), taking the vector  $a$  such that  $\text{ric}(u_0)a = K_2(x)a$  yields

$$Spect(\mathcal{L}_T^x) \leq 1 + \frac{K_2(x)T}{2} + o(T).$$

The proof of (4.11) is completed.  $\square$

**Corollary 4.3.** *Assume (4.4). In the case where  $\text{Ric} = -K_1 \text{Id}$  with  $K_1 \geq 0$ , we have*

$$\left|Spect(\mathcal{L}_T^x) - 1 + \frac{K_1 T}{2}\right| = o(T) \quad \text{as } T \rightarrow 0.$$

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